

# Schwinger-Dyson approach to non-equilibrium classical field theory

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## Abstract

In this paper we discuss a Schwinger-Dyson [SD] approach for determining the time evolution of the unequal time correlation functions of a non-equilibrium classical field theory, where the classical system is described by an initial density matrix at time  $t = 0$ . We focus on  $\lambda\phi^4$  field theory in 1+1 space time dimensions where we can perform exact numerical simulations by sampling an ensemble of initial conditions specified by the initial density matrix. We discuss two approaches. The first, the bare vertex approximation [BVA], is based on ignoring vertex corrections to the SD equations in the auxiliary field formalism relevant for  $1/N$  expansions. The second approximation is a related approximation made to the SD equations of the original formulation in terms of  $\phi$  alone. We compare these SD approximations as well as a Hartree approximation with exact numerical simulations. We find that both approximations based on the SD equations yield good agreement with exact numerical simulations and cure the late time oscillation problem of the Hartree approximation. We also discuss the relationship between the quantum and classical SD equations.

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## I. INTRODUCTION

These past few years, there has been a concerted effort to find approximation schemes for time evolution problems that go beyond mean field theory, such as the Hartree[1, 2] or the leading order in  $1/N$  approximations[3]. Although these approximations are able to give a reasonable picture of the phase diagram of quantum field theories (see for example[4]), there are no true hard scatterings in mean field theory, so there is no mechanism for a system which is out of equilibrium to be driven back into equilibrium. Thus various phenomena found in these early simulations, such as production of disoriented chiral condensates [DCC's], and distortion of the low momentum spectra away from that of thermal equilibrium, might very well be modified when hard scatterings are taken into account. This is especially true in the phenomenological  $O(4)$  sigma model for the chiral phase transition where the effective low energy couplings are large. In Relativistic Heavy Ion Collider [RHIC] physics applications it is important to know, given an expanding plasma, how the time scale for equilibration compares with the time scale for the expansion of the plasma in order to decide whether effects found in the mean field approximation persist following the chiral phase transition.

Initial attempts to go beyond mean field theory, such as truncating the coupled hierarchy of  $n$ -point functions at the 4- or 6-point function level[5, 6], as well as naive use of the  $1/N$  expansion[7, 8], have suffered from serious drawbacks such as negative definite probability and/or secular behavior. These problems, although present in naive perturbation theory, were not present in the mean field approximation because the mean field approximation can be shown to be equivalent to a Hamiltonian dynamical system. Because of these failures, recent attention has been shifted to resummation methods based on Schwinger-Dyson [SD] equations resulting from keeping leading terms in the series for the two particle irreducible [2-PI] generating functional that occurs in the  $\Phi$  derivable approach of Baym[9], or equivalently the formalism of Cornwall, Jackiw, and Tomboulis [CJT] [10, 11, 12, 13]. In this paper we will restrict ourselves to looking at two proposed resummation methods based on the 2-PI approach. The first approach, is a direct use of the loop expansion for the 2-PI generating functional espoused by CJT. This approach has been used to study thermalization in the quantum version of the problem discussed here by Berges and Cox[14]. The second method is to first obtain the exact SD equations in the auxiliary field formalism[15], and then to make the assumption that one can ignore vertex corrections. This approximation also

can be obtained from the 2-PI formalism when we include propagators for the auxiliary field. This latter method we recently applied successfully to the quantum roll problem in 0+1 dimensions[16], where we found that the approximation is a resummation of the next-to-leading order  $1/N$  expansion, which cures the defects of the large- $N$  expansion in next-to-leading order.

In this paper we will apply these approximations to a field theory problem where exact calculations can be carried out. Namely we will consider the time evolution of an initially Gaussian ensemble of classical fields in 1+1 dimensions. We will compare the results of the SD equation approximations with the exact simulation as well as with the Hartree approximation. To make contact with earlier work in this area by Aarts, Bonini, and Wetterich[6], which went beyond Hartree approximation by truncating the hierarchy of Green functions at the four point function level, we will calculate the same variables used by those authors as well as use their initial conditions. Since the quantum evolution in the high temperature limit reduces to this classical field theory problem, by showing that the bare vertex approximation [BVA] gives reasonable quantitative agreement for the classical evolution we guarantee that the BVA gives reasonable results for the *quantum* evolution in the high temperature regime.

The BVA approximation we espouse here is certainly not new. It was first introduced by Kraichnan[17] as the “direct interaction approximation,” which he used to study classical and quantum dynamical systems. What Kraichnan showed was that this approximation, like the Hartree, was realizable as a dynamical system and thus was free from the secularity problem as well as having a positive definite density matrix. A method of deriving these *classical* SD equations was obtained in the classic paper of Martin, Siggia and Rose [MSR] [18]. Although quantum field theorists are now familiar with the Schwinger-Keldysh closed time path [CTP] formalism for initial value problems[19], the related formalism of MSR is less familiar, and so we devote an appendix to deriving the exact SD equations for both *classical* as well as quantum dynamical systems and discuss the connection between the CTP and MSR formalisms. A more extensive discussion of this connection has just been published[20].

This paper is organized as follows. In section II we discuss the classical field theory simulation we are using as a benchmark for various approximations. In section III we review the Hartree approximation. In section IV, we discuss the BVA from various perspectives and compare the quantum and classical versions. In section V, we discuss the 2-PI approximation

used by Berges and Cox, and show that it is a re-expansion of the BVA. In section VI we compare all three approximation schemes with the exact numerical solutions. We conclude in section VII.

In addition, in the appendices we discuss both the CTP and MSR formalisms for a problem with two interacting fields and discuss the relationship between the quantum and classical SD equations.

## II. CLASSICAL SCALAR FIELD THEORY

### A. The Lagrangian

We are interested in studying the classical scalar field theory in 1+1 dimensions described by the Lagrangian density:

$$\mathcal{L} = \frac{1}{2} [(\partial_t \phi)^2 - (\partial_x \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{8} \phi^4, \quad (1)$$

with the equation of motion:

$$[\partial_t^2 - \partial_x^2 + \mu^2] \phi(x, t) + \frac{\lambda}{2} \phi^3(x, t) = 0. \quad (2)$$

In momentum space, we define:

$$\phi(x, t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{\phi}_k(t) e^{ikx}, \quad (3)$$

in which case (2) becomes:

$$[\partial_t^2 + \omega_k^2(t)] \tilde{\phi}_k(t) + \frac{\lambda}{2} \int_{-\infty}^{+\infty} dx \phi^3(x, t) e^{-ikx} = 0, \quad (4)$$

where  $\omega_k^2 = k^2 + \mu^2$ . Because we are particularly interested in SD equations which are a re-summation of the large- $N$  expansion, we will also be using a second form of this Lagrangian:

$$\mathcal{L} = \frac{1}{2} [(\partial_t \phi)^2 - (\partial_x \phi)^2 - \chi \phi^2] + \frac{\chi}{\lambda} \left( \frac{\chi}{2} - \mu^2 \right). \quad (5)$$

This second Lagrangian leads to the equations of motion:

$$[\partial_t^2 - \partial_x^2 + \chi(x, t)] \phi(x, t) = 0, \quad (6)$$

and the constraint (“gap”) equation for  $\chi(x, t)$ :

$$\chi(x, t) = \mu^2 + \frac{\lambda}{2} \phi^2(x, t). \quad (7)$$

The Hamiltonian density is given by:

$$\mathcal{H} = \frac{1}{2} [(\partial_t \phi)^2 + (\partial_x \phi)^2 + \chi \phi^2] - \frac{\chi}{\lambda} \left( \frac{\chi}{2} - \mu^2 \right). \quad (8)$$

## B. Initial ensemble

Solutions of the equations of motion are found for the initial conditions  $\phi(x, 0) = \phi(x)$  and  $\pi(x, 0) = \pi(x)$ . If the system is in thermal equilibrium, the initial values  $\phi(x)$  and  $\pi(x)$  are taken from a canonical ensemble governed by a classical density distribution  $\rho[\phi, \pi]$ , given by:

$$\begin{aligned} \rho[\phi, \pi] &= Z^{-1}(\beta) \exp\{-\beta H[\phi, \pi]\}, \\ Z(\beta) &= \prod_x \int \mathrm{d}\phi(x) \mathrm{d}\pi(x) \exp\{-\beta H[\phi, \pi]\}. \end{aligned} \quad (9)$$

with  $\beta = 1/T$ . The ensemble average of a quantity  $A[\phi, \pi]$  is then defined by:

$$\begin{aligned} \langle A[\phi, \pi] \rangle &\equiv \mathrm{Tr}\{\rho A[\phi, \pi]\} \\ &\equiv Z^{-1}(\beta) \prod_x \int \mathrm{d}\phi(x) \mathrm{d}\pi(x) A[\phi, \pi] \exp\{-\beta H[\phi, \pi]\}, \end{aligned} \quad (10)$$

In order to make direct comparisons with the work of Aarts, Bonini and Wetterich[5], we choose an ensemble constructed from unperturbed Hamiltonian  $H_0$  rather than  $H$  in this paper. This is *not* a canonical equilibrium distribution for either the exact solution *or* the Hartree approximation, so it gives a non-trivial initial value problem for all our approximations. In classical 1+1 dimensional field theory, mass renormalization is finite so that one can use any arbitrary mass in our initial  $H_0$ ; however, in order to compare our results with Aarts, *et. al.*, we take this mass to be the bare (unperturbed) mass  $\mu$ . In dimensions higher than 1+1 one would need to use a renormalized mass parameter in  $H_0$ . So we choose for our initial density matrix:

$$\rho_0[\phi_0, \pi_0] = Z_0^{-1}(\beta_0) \exp\{-\beta_0 H_0[\phi_0, \pi_0]\}, \quad (11)$$

where  $\beta_0 = 1/T_0$  is to be regarded as a parameter. The unperturbed Hamiltonian is

$$H_0[\phi_0, \pi_0] = \frac{1}{2} \int_{-\infty}^{+\infty} \mathrm{d}x [\pi_0^2(x) + (\partial_x \phi_0(x))^2 + \mu^2 \phi_0^2(x)], \quad (12)$$

with the equations of motion,

$$[\partial_t^2 - \partial_x^2 + \mu^2] \phi_0(x, t) = 0, \quad (13)$$

where  $\pi_0(x, t) = \dot{\phi}_0(x, t)$ . Again, defining Fourier transforms by:

$$\phi(x, t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{\phi}_k(t) e^{ikx}, \quad \pi(x, t) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \tilde{\pi}_k(t) e^{ikx},$$

solutions for the free particle case are given by:

$$\begin{aligned} \tilde{\phi}_{0k}(t) &= \phi_k \cos(\omega_k t) + (\pi_k / \omega_k) \sin(\omega_k t), \\ &= \frac{1}{\sqrt{2\omega_k}} \left\{ a_k e^{-i\omega_k t} + a_{-k}^* e^{i\omega_k t} \right\}, \\ \tilde{\pi}_{0k}(t) &= \pi_k \cos(\omega_k t) - \phi_k \omega_k \sin(\omega_k t), \\ &= \frac{1}{i} \sqrt{\frac{\omega_k}{2}} \left\{ a_k e^{-i\omega_k t} - a_{-k}^* e^{i\omega_k t} \right\}, \end{aligned} \quad (14)$$

where  $\omega_k = \sqrt{k^2 + \mu^2}$ . Note that  $\phi(x, 0) = \phi_0(x, 0)$  and  $\pi(x, 0) = \pi_0(x, 0)$ . So then the density matrix becomes:

$$\rho_0[x_k, y_k] = \frac{1}{Z_0(\beta_0)} \exp \left\{ -\beta_0 \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \omega_k (x_k^2 + y_k^2) \right\}. \quad (15)$$

where we have put  $a_k = x_k + iy_k$ . So for the Monte Carlo calculation, we select value  $x_k$  and  $y_k$  from the Gaussian distribution in Eq. (15), and use these values to construct starting values for  $\phi_0(x, 0)$  and  $\pi_0(x, 0)$ . These functions are not smooth functions of  $x$ . From (15), we find that:

$$\begin{aligned} \langle a_k^* a_{k'} \rangle_0 &= \langle a_{k'} a_k^* \rangle_0 = 2\pi \delta(k - k') n_k(\beta_0), \\ \langle a_k a_{k'} \rangle_0 &= \langle a_k^* a_{k'}^* \rangle_0 = 0, \end{aligned} \quad (16)$$

and where  $n_k(\beta_0) = 1/(\beta_0 \omega_k)$ . Note that  $n_k(\beta_0)$  is the high temperature limit of the classical Bose-Einstein occupation number distribution. Then, one show that  $\langle \phi(x, 0) \rangle_0 = \langle \pi(x, 0) \rangle_0 = 0$ , and that

$$\phi_{\text{cl}}^2(0) = \frac{1}{L} \int_0^L dx \langle \phi^2(x, 0) \rangle_0 = \int_{-\Lambda}^{+\Lambda} \frac{dk}{2\pi} \frac{n_k(\beta_0)}{\omega_k} = \frac{I(\Lambda)}{2\mu\beta_0}, \quad (17)$$

$$\pi_{\text{cl}}^2(0) = \frac{1}{L} \int_0^L dx \langle \pi^2(x, 0) \rangle_0 = \int_{-\Lambda}^{+\Lambda} \frac{dk}{2\pi} n_k(\beta_0) \omega_k = \frac{2\mu J(\Lambda)}{\beta_0}, \quad (18)$$

where we have introduced cutoffs  $L$  and  $\Lambda$  in coordinate and momentum space integrals. We have then:

$$\begin{aligned} I(\Lambda) &= 2\mu \int_{-\Lambda}^{\Lambda} \frac{dk}{2\pi} \frac{1}{\omega_k^2} = \frac{2}{\pi} \tan^{-1} \left( \frac{\Lambda}{\mu} \right) \sim 1 - \frac{2\mu}{\pi\Lambda} + \dots, \\ J(\Lambda) &= \frac{1}{2\mu} \int_{-\Lambda}^{+\Lambda} \frac{dk}{2\pi} = \frac{\Lambda}{\pi 2\mu}. \end{aligned} \quad (19)$$

### C. Correlation and Green functions

We define correlation functions and Green functions by ensemble averages:

$$\begin{aligned} F(x, x') &= \langle \phi(x) \phi(x') \rangle_0, \\ G_R(x, x') &= -\sigma(x, x') \Theta(t - t'), \\ G_A(x, x') &= +\sigma(x, x') \Theta(t' - t), \end{aligned} \quad (20)$$

where  $\sigma(x, x')$  is the spectral function, given by the ensemble average of the classical Poisson bracket of  $\phi(x)$  with  $\phi(x')$ :

$$\sigma(x, x') = \langle \{ \phi(x), \phi(x') \} \rangle_0. \quad (21)$$

[In this section, we use  $x \equiv (x, t)$  when needed.] Since  $\rho_0[\phi_0, \pi_0]$  is Gaussian, the only nontrivial correlation and Green functions for the free particle case at  $t = 0$  are the two-point functions. These functions are discussed in detail in Ref. [21]. Writing  $\phi_0(x)$  in a Fourier expansion and using Eqs. (16), the unperturbed correlation and structure functions are given by:

$$\begin{aligned} F_0(x, x') &\equiv \langle \phi_0(x) \phi_0(x') \rangle_0 = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} n_k(\beta_0) \frac{\cos[\omega_k(t - t')]}{\omega_k} e^{ik(x-x')}, \\ \sigma_0(x, x') &\equiv \langle \{ \phi_0(x), \phi_0(x') \} \rangle_0 = - \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{\sin[\omega_k(t - t')]}{\omega_k} e^{ik(x-x')}. \end{aligned} \quad (22)$$

There are similar correlation and Green functions for the  $\chi(x)$  field. Even though this field is a constraint field, we treat it as if it is dynamic and then take a limit so that the free particle Green functions, which in this case we call  $D_{0R}(x, x')$  and  $D_{0A}(x, x')$ , become just delta functions.

#### D. Classical solutions on the lattice

We introduce a lattice in coordinate and momentum space by putting  $x \rightarrow x_n$  and  $k \rightarrow k_m$ , where:

$$\begin{aligned} x_n &= n a , & n &= 0, 1, \dots, N , \\ k_m &= 2\pi m/L , & m &= -N/2, -N/2 + 1, \dots, N/2 - 1 , \end{aligned} \quad (23)$$

where that  $L = Na$  and the momentum cutoff  $\Lambda = \pi/a$ . We write  $\phi_n(t) \equiv \phi(x_n, t)$  and  $\tilde{\phi}_{k_m}(t) \equiv \tilde{\phi}_m(t)$ . Fourier transform relations become:

$$\phi_n(t) = \frac{1}{L} \sum_{m=-N/2}^{N/2-1} \tilde{\phi}_m(t) e^{2\pi i n m/N} , \quad \tilde{\phi}_m(t) = a \sum_{n=0}^{N-1} \phi_n(t) e^{-2\pi i n m/N} .$$

There are two strategies we can use to step out solutions. In coordinate space, we can introduce a time grid,  $t_s = sa_0$ ,  $s = 0, 1, \dots$ , and replace the differential operators for space and time by second order difference formulas:

$$\begin{aligned} \partial_x^2 \phi(x, t) &\rightarrow \left[ \phi_{n+1,s} - 2\phi_{n,s} + \phi_{n-1,s} \right] / a^2 , \\ \partial_t^2 \phi(x, t) &\rightarrow \left[ \phi_{m,s+1} - 2\phi_{m,s} + \phi_{m,s-1} \right] / a_0^2 , \end{aligned} \quad (24)$$

So that Eq. (2) becomes:

$$\begin{aligned} \phi_{m,s+1} &= 2\phi_{m,s} - \phi_{m,s-1} - a_0^2 \left\{ \mu^2 \phi_{m,s} + \frac{\lambda}{2} \phi_{m,s}^3 \right\} \\ &\quad + \left( \frac{a_0}{a} \right)^2 \left\{ \phi_{n+1,s} - 2\phi_{n,s} + \phi_{n-1,s} \right\} . \end{aligned} \quad (25)$$

This method is called a “staggered leapfrog” approximation in the literature[32], and is believed to be stable. The advantage of this coordinate space method on the lattice is that the interaction term,  $\phi^3(x, t)$ , is easy to calculate. We construct an average of all Monte Carlo runs  $M$  for all values of  $x_n$  and compute:

$$\phi_{\text{cl}}^2(t_s) = \frac{1}{MN} \sum_{i=1}^M \sum_{n=0}^{N-1} \phi_{n,s}^2 \sim \frac{1}{L} \int_0^L dx \langle \phi^2(x, t_s) \rangle_0 . \quad (26)$$

The use of the second order difference formula means that the dispersion relation is now given by:

$$\omega_k^2 \rightarrow \hat{\omega}_m^2 = \hat{k}_m^2 + \mu^2 , \quad \hat{k}_m = \frac{2}{a} \sin \left( \frac{\pi m}{N} \right) . \quad (27)$$



So at  $t = 0$ ,  $\langle \phi^2(x, 0) \rangle_0$  is still given by Eq. (17) but  $I(\Lambda)$  is now given by:

$$I(\Lambda) = \frac{2\mu}{L} \sum_{m=-N/2}^{N/2-1} \frac{1}{\hat{\omega}_m^2} = \frac{1}{\sqrt{1 + (\pi\mu/2\Lambda)^2}} \sim 1 - \frac{1}{2} \left( \frac{\pi\mu}{2\Lambda} \right)^2 + \dots \quad (28)$$

We call this method “coordinate-space Monte Carlo.”

The second strategy, is to solve Eq. (4) in momentum space, and use a fast Fourier transform to compute the convolution integral. Any time integrator can be used for the time step, but we employ the same second order finite difference formula as above. Here, there is no use of the second order difference formula, so the dispersion relation is given by:

$$\omega_k^2 \rightarrow \omega_m^2 = k_m^2 + \mu^2, \quad k_m = \frac{2\pi m}{L}. \quad (29)$$

$\langle \phi^2(x, 0) \rangle_0$  is again given by Eq. (17) but  $I(\Lambda)$  is now given by:

$$\begin{aligned} I(\Lambda) &= \frac{2\mu}{L} \sum_{m=-N/2}^{N/2-1} \frac{1}{\omega_m^2} = \left( \frac{\pi\mu N}{2\Lambda} \right) \sum_{m=-N/2}^{N/2-1} \frac{1}{(\pi m)^2 + (\pi\mu N/2\Lambda)^2} \\ &\sim \frac{2}{\pi} \tan^{-1} \left( \frac{\Lambda}{\mu} \right), \end{aligned} \quad (30)$$

for sufficiently large  $N$ .

We call this method “momentum-space Monte Carlo.” Since our BVA codes use a global-in-time Chebyshev expansion method in momentum space, with a dispersion relation given by Eq. (29), we can compare directly the results of the BVA code to the classical momentum-space Monte Carlo calculations.

### III. HARTREE APPROXIMATION

The Hartree approximation is a simple truncation scheme which is obtained by setting all correlation functions beyond the second one to zero. Since it has a simple interpretation in terms of a time-dependent variational approximation where the density matrix is assumed to be Gaussian, it automatically corresponds to a Hamiltonian dynamical system so it is free from unphysical behavior. Here we discuss the classical version of this approximation. We are interested in ensemble averages of the classical evolution. The equation for the expectation value  $\phi_c(x) = \langle \phi(x) \rangle_0$  is:

$$\left[ \partial_t^2 - \partial_x^2 + \mu^2 \right] \phi_c(x) + \frac{\lambda}{2} \langle \phi^3(x) \rangle_0 = 0. \quad (31)$$

[Here  $x \equiv (x, t)$ .] Since the Hartree approximation is equivalent to an initial Gaussian ensemble staying Gaussian, the expectation values obey the factorization condition

$$\langle \phi^3(x) \rangle_0 = \phi_c^3(x) + 3 G(x, x) \phi_c(x)$$

where

$$G(x, x') = \langle \phi(x) \phi(x') \rangle_0 - \langle \phi(x) \rangle_0 \langle \phi(x') \rangle_0.$$

This equation must be supplemented by the equation for the expectation value of the fluctuation, which under the Gaussian hypothesis obeys:

$$\left\{ \partial_t^2 - \partial_x^2 + \mu^2 + \frac{3\lambda}{2} \left\{ \phi_c^2(x) + G(x, x) \right\} \right\} G(x, x') = 0. \quad (32)$$

We will determine the 2-point function by assuming that we can write a mode decomposition for  $\phi(x, t)$  of the form:

$$\phi(x) = \phi_c(x) + \int \frac{dk}{2\pi} [a_k f_k(x) + a_k^* f_k^*(x)] \quad (33)$$

where the complete orthonormal set of mode functions  $f_k(x)$  obey

$$\left\{ \partial_t^2 - \partial_x^2 + \mu^2 + \frac{3\lambda}{2} \left\{ \phi_c^2(x) + G(x, x) \right\} \right\} f_k(x) = 0. \quad (34)$$

The ensemble average of all the bilinears of  $a_k$  and  $a_k^*$  are determined by the initial ensemble average. For our classical evolution we have

$$\langle a_k^* a_{k'} \rangle_0 = \langle a_{k'} a_k^* \rangle_0 = 2\pi n_k(\beta_0) \delta(k - k'). \quad (35)$$

where, again,  $n_k(\beta_0) = 1/(\beta_0 \omega_k)$ , with  $\omega_k = \sqrt{k^2 + \mu^2}$ .

We also need to specify the initial values of  $f_k(x)$  and  $\dot{f}_k(x)$  to complete the calculation. These are determined from the initial averages given in Eq. (16) and  $\langle \phi \phi \rangle_0$ ,  $\langle \pi \pi \rangle_0$ , and  $\langle \phi \pi \rangle_0$  at  $t = 0$ . We find:

$$f_k(0) = \frac{e^{ikx}}{\sqrt{2\omega_k}}, \quad \dot{f}_k(0) = -i\omega_k f_k(0). \quad (36)$$

To summarize, for the translationally invariant case when  $\phi_c(x, t)$  is independent of  $x$ , the equations that need to be solved simultaneously are:

$$\begin{aligned} [\partial_t^2 + \chi_1(t)] \phi_c(t) &= 0, & [\partial_t^2 + k^2 + \chi_2(t)] f_k(t) &= 0, \\ \chi_1(t) &= \mu^2 + \frac{\lambda}{2} \phi_c^2(t) + \frac{3\lambda}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} 2n_k(\beta_0) |f_k(t)|^2, \\ \chi_2(t) &= \mu^2 + \frac{3\lambda}{2} \phi_c^2(t) + \frac{3\lambda}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} 2n_k(\beta_0) |f_k(t)|^2. \end{aligned} \quad (37)$$

The ensemble average of the fields is given by:

$$\langle \phi^2(t) \rangle_0 = \phi_c^2(t) + \int_{-\infty}^{+\infty} \frac{dk}{2\pi} 2n_k(\beta_0) |f_k(t)|^2. \quad (38)$$

We remark here that, with the replacement  $2n_k + 1 \rightarrow 2n_k$ , the evolution equations for the Hartree approximation in quantum mechanics is the same as the classical ones. In addition, in the quantum case, in thermal equilibrium,  $n_k$  is given by a Bose-Einstein occupation distribution. Quantum mechanics also affects the initial conditions of the mode functions, whose Wronskian, in the quantum case, must yield  $i\hbar$ .

For completeness we mention here that in the quantum case, the effective action for the Hartree approximation is

$$S_{\text{eff}} = \int dx \mathcal{L}_{\text{cl}}(\phi, \chi_2) + \frac{i}{2} \text{Tr}[\ln(\square + \chi_2)], \quad (39)$$

where  $\mathcal{L}_{\text{cl}}(\phi, \chi_2)$  is the classical action written in terms of the auxiliary field  $\chi_2$ , given by:

$$\begin{aligned} \mathcal{L}_{\text{cl}}(\phi, \chi_2) &= -\frac{1}{2} \phi(x) [\square + \chi_2(x)] \phi(x) + \frac{\lambda}{4} \phi^4(x) + \frac{1}{3\lambda} \left[ \frac{\chi_2^2(x)}{2} - \mu^2 \chi_2(x) \right], \\ \chi_2(x) &= \mu^2 + \frac{3\lambda}{2} \phi^2(x), \end{aligned}$$

Here the 3 in the Hartree approximation occurring in front of  $\lambda/2$  would become  $1 + \frac{2}{N}$  when there are  $N$  fields, showing that the Hartree reduces to the large- $N$  approximation at large  $N$ .

We notice that unlike the case of the large- $N$  expansion, the classical Lagrangian still has a quartic term which has to be treated as an *external* classical field when doing the remaining quadratic path integral over  $\phi$  to obtain the above determinant term in the effective action. Thus, when we take expectation values of the Lagrangian to obtain the energy momentum tensor, the quartic term only gets a classical contribution and does not give fluctuation contributions. In taking the expectation value in the Hartree approximation one treats  $\chi$  as well as the quartic term in  $\phi$  classically.

In terms of the mode functions, the energy density is given by:

$$\begin{aligned} \varepsilon &= \frac{1}{2} [(\partial_t \phi^c)^2 + (\partial_x \phi^c)^2 + \chi_2 \phi_c^2] - \frac{1}{3\lambda} \left[ \frac{\chi_2^2(x)}{2} - \mu^2 \chi_2(x) \right] - \frac{\lambda}{4} \phi_c^4 \\ &\quad + \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} 2n_k(\beta_0) [|f_k(t)|^2 + (k^2 + \chi_2) |f_k|^2] \end{aligned} \quad (40)$$

Because classically  $n_k \omega_k = T_0$  represents the equipartition of energy, the classical energy density suffers a linear divergence. In the quantum case, the Bose-Einstein distribution cures this, and instead one obtains a quadratic divergence from the zero point quantum fluctuations, which must be renormalized by a constant counterterm, corresponding to a cosmological constant.

#### IV. BARE VERTEX APPROXIMATION

The Hartree approximation truncates what would be otherwise an infinite set of coupled equations for the expectation values of products of the classical field by assuming that all fluctuations higher than the second one are exactly zero. In quantum mechanical applications, this approximation has many nice features such as being derivable from a variational approximation and having a well defined positive definite density function at all times. However, it suffers from not containing the hard scatterings usually associated with thermalization. It is thus important to find approximations which include scattering and which are free from the problems of secularity and positivity as we discussed in the introduction.

From the generating functional of all the correlation functions one finds that the Green functions for both quantum and classical field theory obey a set of coupled integral equations relating the exact two point functions to themselves and higher order one-particle irreducible vertex functions. In quantum field theory, the easiest way to derive these is by using a generating functional for the matrix Green functions of the CTP formalism. The classical theory has similar structure to the Green functions found when we tridiagonalize the matrix Green functions and write SD equations for the retarded Green functions and Wightman functions. The classical limit ( $\hbar \rightarrow 0$ ) of the BVA is obtained by using the classical value for the free retarded Green functions and Wightman function and also only keeping those graphs which are leading at high temperature and small coupling as discussed by Aarts and Smit[22] and by Buchmüller and Jakovác[23, 24]. For the classical theory, one can define free propagators which replace the quantum ones by replacing commutators by Poisson brackets, as discussed by Parisi[21]. In appendix B, we show how to make precise the connection between the exact quantum and classical Schwinger-Dyson equations, by comparing the CTP formalism with the MSR formalism. We find that certain bare vertices are missing in the classical case.

The BVA is an approximation in which the exact two point functions are used in the equations for  $\langle \phi(x) \rangle_0$  and  $\langle \phi(x)\phi(x') \rangle_0$ , but the exact  $\langle \phi(x)\phi(x')\chi(x'') \rangle_0$  vertex function is replaced by the bare one. This approximation was discussed from several points of view in our previous paper[16], where we showed that the BVA is a particular resummation of the  $1/N$  expansion. For the classical problem we treat here, the exact SD equations are obtained from the MSR formalism[18]. We sketch the details of this in appendix B. What is important about the BVA is that this approximation is energy conserving, is nonsecular and, as Kraichnan has shown[17], the effective Hamiltonian has positive spectra.

In condensed matter physics the BVA is used in the electron-phonon problem for the description of the normal state of a Fermi liquid. In this case the first vertex correction to the self-energy is small compared to the bare vertex and are of the order of  $m_e/M_{ion}$  where  $m_e$  is the electron mass and  $M_{ion}$  is the ion mass. This result is known as the Migdal theorem[25]. Close to a superconducting instability the theorem is not valid and a partial resummation of the vertex corrections is necessary[26]. The physical reason for the validity of Migdal's theorem is the adiabatic motion of the electronic degrees of freedom compared to the ionic degrees of freedom. In quantum field theory it is not yet clear what the domain of validity of the BVA will be. Here we are concerned with its validity in the high temperature limit.

### A. The quantum Schwinger-Dyson equations in the BVA

The exact SD equations for  $\lambda\phi^4$  field theory, rewritten in terms of an auxiliary field, were first derived by Bender, Cooper and Guralnik[15] and discussed in detail in the CTP formalism in Ref. [16]. We will also derive the SD equation for both the classical and quantum cases in the appendix. The CTP formalism and notation is reviewed in appendix A. For the case where  $\phi(x) = \langle \hat{\phi}(x) \rangle_0 = 0$ , one has that

$$D(x, x') = D_0(x, x') - \int_{\mathcal{C}} d^2x_1 \int_{\mathcal{C}} d^2x_2 D_0(x, x_1) \Pi(x_1, x_2) D(x_2, x'), \quad (41)$$

$$G(x, x') = G_0(x, x') - \int_{\mathcal{C}} d^2x_1 \int_{\mathcal{C}} d^2x_2 G_0(x, x_1) \Sigma(x_1, x_2) G(x_2, x'), \quad (42)$$

The gap equation for  $\chi(t)$  is:

$$\chi(t) = \mu^2 + \frac{\lambda}{2} G(x, x)/i. \quad (43)$$

Here  $D_0^{-1}(x, x')$  and  $G_0^{-1}(x, x')$  are given by:

$$D_0^{-1}(x, x') = -\frac{1}{\lambda} \delta_{\mathcal{C}}(x, x') , \quad (44)$$

$$G_0^{-1}(x, x') = [\square + \chi(t)] \delta_{\mathcal{C}}(x, x') . \quad (45)$$

Thus  $D_0(x, x') = -\lambda \delta_{\mathcal{C}}(x, x')$ . The exact equations for the polarization  $\Pi$  and the self energy  $\Sigma$  are

$$\Pi(x, x') = \frac{i}{2} \int dx_1 dx_2 G(x, x_1) \Gamma(x_1, x', x_2) G(x_2, x) , \quad (46)$$

$$\Sigma(x, x') = i \int dx_1 dx_2 G(x, x_1) \Gamma(x_1, x', x_2) D(x_2, x) . \quad (47)$$

In the BVA we make the further approximation that the *exact* one-particle irreducible  $\chi\phi\phi$  vertex function  $\Gamma$  is replaced by its lowest order value.

$$\Gamma(x_1, x', x_2) = \delta(x_1 - x') \delta(x_2 - x') . \quad (48)$$

so that we have in the quantum BVA

$$\Pi(x, x') = \frac{i}{2} G(x, x') G(x', x) , \quad \Sigma(x, x') = i G(x, x') D(x', x) . \quad (49)$$

To remove the tadpole contributions we write

$$D(x, x') = D_0(x, x') + \bar{D}(x, x') = -\lambda \delta_{\mathcal{C}}(x, x') + \bar{D}(x, x') , \quad (50)$$

which we put into Eq. (41), to find an integral equation for  $\bar{D}(x, x')$ :

$$\bar{D}(x, x') = -\lambda^2 \Pi(x, x') + \lambda \int_{\mathcal{C}} d^2 x'' \Pi(x, x'') \bar{D}(x'', x') . \quad (51)$$

In addition, we put

$$\Sigma(x, x') = \lambda [G(x, x)/i] \delta_{\mathcal{C}}(x, x') + \bar{\Sigma}(x, x') , \quad (52)$$

where

$$\bar{\Sigma}(x, x') = i G(x, x') \bar{D}(x', x) . \quad (53)$$

Now by multiplying Eq. (42) by  $G_0^{-1}(x, x')$  gives a differential-integral equation for  $G(x, x')$ :

$$[\square + \chi(t)] G(x, x') = \delta_{\mathcal{C}}(x, x') - \int_{\mathcal{C}} d^2 x'' \Sigma(x, x'') G(x'', x') , \quad (54)$$

Putting (52) into (54) gives:

$$[\square + \bar{\chi}(t)] G(x, x') = \delta_{\mathcal{C}}(x, x') - \int_{\mathcal{C}} d^2 x'' \bar{\Sigma}(x, x'') G(x'', x'), \quad (55)$$

where we define  $\bar{\chi}(t)$  by

$$\bar{\chi}(t) = \chi(t) + \lambda G(x, x)/i = \mu^2 + \frac{3\lambda}{2} G(x, x)/i, \quad (56)$$

which is the Hartree approximation effective mass. In summary the full set of equations are

$$[\square + \bar{\chi}(t)] G(x, x') = \delta_{\mathcal{C}}(x, x') - \int_{\mathcal{C}} d^2 x'' \bar{\Sigma}(x, x'') G(x'', x'), \quad (57)$$

$$\bar{D}(x, x') = -\lambda^2 \Pi(x, x') + \lambda \int_{\mathcal{C}} d^2 x'' \Pi(x, x'') \bar{D}(x'', x'). \quad (58)$$

$$\bar{\Sigma}(x, x') = i G(x, x') \bar{D}(x', x). \quad (59)$$

$$\Pi(x, x') = \frac{i}{2} G(x, x') G(x', x), \quad (60)$$

$$\bar{\chi}(t) = \mu^2 + \frac{3\lambda}{2} G(x, x)/i. \quad (61)$$

Note the factor of 3/2 in Eq. (61). We can now *redefine*  $G_0(x, x')$  using  $\bar{\chi}(t)$  rather than  $\chi(t)$ . We call this propagator  $\bar{G}_0(x, x')$ . So let

$$\bar{G}_0^{-1}(x, x') = [\square + \bar{\chi}(t)] \delta_{\mathcal{C}}(x, x'). \quad (62)$$

Then Eq. (57) can be written as an integral equation for  $G(x, x')$  using  $\bar{G}_0(x, x')$  and  $\bar{\Sigma}(x, x')$ :

$$G(x, x') = \bar{G}_0(x, x') - \int_{\mathcal{C}} d^2 x_1 \int_{\mathcal{C}} d^2 x_2 \bar{G}_0(x, x_1) \bar{\Sigma}(x_1, x_2) G(x_2, x'), \quad (63)$$

which is completely equivalent to the original equations, except that now we have explicitly removed the delta-function term in  $D(x, x')$  and included it in the definition of  $\bar{\chi}(t)$ .

## B. Two-particle irreducible effective action

As we have discussed in a previous paper, the BVA can also be obtained by keeping the two loop graph in the effective action for the generating functional for the 2-PI graphs[10]. Namely, the effective action is the twice-Legendre transformed generating functional:

$$\Gamma[\Phi, \mathcal{G}] = S_{\text{class}}[\Phi] + \frac{i}{2} \text{Tr}\{\ln[\mathcal{G}^{-1}]\} + \frac{i}{2} \text{Tr}\{G^{-1}[\Phi] \mathcal{G} - 1\} + \Gamma_2[\Phi, \mathcal{G}]. \quad (64)$$

where

$$G_{\alpha,\beta}^{-1}[\Phi](x, x') = -\frac{\delta^2 S_{\text{cl}}[\Phi]}{\delta\Phi_\alpha(x) \delta\Phi_\beta(x')} \quad (65)$$

For our problem  $\Phi$  consists of both  $\phi$  and  $\chi$  and is also a matrix in CTP space.  $S_{\text{cl}}$  is the classical Lagrangian written in terms of both  $\phi$  and  $\chi$ . To obtain the BVA one just keeps the two loop graph made from three  $\Phi$  propagators. For the case  $\langle\phi\rangle \neq 0$ , the  $\Phi$  propagator will be non diagonal in  $\phi$ - $\chi$  space. The quantity  $\Gamma_2[\Phi, \mathcal{G}]$  has a simple graphical interpretation in terms of all the 2-PI vacuum graphs using vertices from the interaction term. When  $\langle\phi\rangle_0 = 0$ , one obtains

$$\begin{aligned} \Gamma[\chi, \mathcal{G}_\phi, \mathcal{D}] &= S_{\text{class}}[\chi] + \frac{i}{2} \text{Tr}\{\ln[\mathcal{D}^{-1}]\} \\ &+ \frac{i}{2} \text{Tr}\{\ln[\mathcal{G}_\phi^{-1}]\} + \frac{i}{2} \text{Tr}\{D^{-1} \mathcal{D} + G^{-1} \mathcal{G}_\phi - 2\} + \Gamma_2[\mathcal{G}_\phi, \mathcal{D}]. \end{aligned} \quad (66)$$

where  $\mathcal{G} \equiv \{\mathcal{G}_\phi, \mathcal{D}\}$  and

$$\Gamma_2[\mathcal{G}_\phi, \mathcal{D}] = -\frac{1}{4} \int_{\mathcal{C}} dx_1 \int_{\mathcal{C}} dx_2 \mathcal{D}(x_1, x_2) \mathcal{G}_\phi(x_1, x_2) \mathcal{G}_\phi(x_2, x_1). \quad (67)$$

It is often useful to put this generating functional in the Baym form by defining  $\Sigma = \mathcal{G}^{-1} - G^{-1}$  to obtain:

$$\Gamma[\Phi, \mathcal{G}] = S_{\text{class}}[\Phi] + \frac{i}{2} \text{Tr}\{\ln[\mathcal{G}^{-1}]\} - \frac{i}{2} \text{Tr}\{\Sigma \mathcal{G}\} + \Gamma_2[\Phi, \mathcal{G}]. \quad (68)$$

Eq. (68) allows us to relate the graphs in the self energy  $\Sigma$  to those kept in the effective action  $\Gamma_2$  by means of

$$\frac{i}{2} \Sigma = \frac{\delta \Gamma_2}{\delta \mathcal{G}}.$$

### C. Classical limit

As we discuss in the appendices, the SD equations for quantum and classical evolutions are similar in structure, the only difference being that the classical evolution contains fewer vertices. The interaction part of the CTP Lagrangian in the original matrix bases is given by

$$L_I = \frac{1}{2} [\chi_+ \phi_+ \phi_+ - \chi_- \phi_- \phi_-] \quad (69)$$



Rotating to the advanced-retarded basis by the transformation

$$\begin{aligned}\chi_+ &= \chi + \frac{\hbar}{2} \hat{\pi}_\chi, & \chi_- &= \chi - \frac{\hbar}{2} \hat{\pi}_\chi, \\ \phi_+ &= \phi + \frac{\hbar}{2} \hat{\pi}_\phi, & \phi_- &= \phi - \frac{\hbar}{2} \hat{\pi}_\phi,\end{aligned}\tag{70}$$

the Lagrangian becomes

$$L_I/\hbar = \frac{1}{2} \left\{ 2\chi\phi\hat{\pi}_\phi + \hat{\pi}_\chi \left[ \phi^2 + \frac{\hbar^2}{4} \hat{\pi}_\phi^2 \right] \right\}\tag{71}$$

so that the bare vertices of the rotated Lagrangian are:

$$\gamma_{\chi\phi\hat{\pi}_\phi} = \gamma_{\phi\chi\hat{\pi}_\phi} = \dots 4 \text{ perms} \dots = \frac{1}{6}, \quad \gamma_{\phi\phi\hat{\pi}_\chi} = \gamma_{\hat{\pi}_\chi\phi\phi} = \gamma_{\phi\hat{\pi}_\chi\phi} = \frac{1}{3},\tag{72}$$

$$\gamma_{\hat{\pi}_\chi\hat{\pi}_\phi\hat{\pi}_\phi} = \gamma_{\hat{\pi}_\phi\hat{\pi}_\chi\hat{\pi}_\phi} = \gamma_{\hat{\pi}_\phi\hat{\pi}_\phi\hat{\pi}_\chi} = \frac{\hbar^2}{24}.\tag{73}$$

In the classical limit the vertices of the last line which have an extra factor of  $\hbar^2$  are missing. The self energy in the BVA is given by

$$\Sigma_{ab} = \gamma_{ajk} G^{jl} D^{km} \gamma_{lmb}.\tag{74}$$

In the above  $j, l$  are summed over  $\phi$  and  $\hat{\pi}_\phi$  whereas  $k, m$  are summed over  $\chi$  and  $\hat{\pi}_\chi$ . Similarly the vacuum polarization in the BVA is given by

$$\Pi_{ab} = \frac{1}{2} \gamma_{ajk} G^{jl} G^{km} \gamma_{lmb},\tag{75}$$

where in the above  $j, l, k, m$  are summed over  $\phi$  and  $\hat{\pi}_\phi$ . The inverse Green's functions are directly related to the self energies as discussed in Appendix B.

## V. THE EFFECTIVE ACTION IN THE SINGLE FIELD FORMULATION

In the single field formulation, we consider an action given by:

$$\Gamma[\phi, G] = S_{\text{class}}[\phi] + \frac{i}{2} \text{Tr}\{\ln[G^{-1}]\} - \frac{i}{2} \text{Tr}\{\Sigma G\} + \Gamma_2[\phi, G].\tag{76}$$

where now  $S_{\text{class}}$  is the action with the usual  $\phi^4$  self-interactions. The Hartree approximation keeps the two-loop contribution to  $\Gamma_2[\phi, G]$  given by:

$$\Gamma_2[\phi, G]_{\text{Hartree}} = \frac{3\lambda}{8} \int dx [G(x, x)]^2.\tag{77}$$

It is easy to go beyond Hartree by using the three-loop contribution:

$$\Gamma_2[\phi, G]_{\text{three-loop}} = \frac{\lambda^2}{16} \iint dx_1 dx_2 [G(x_1, x_2)]^4 \quad (78)$$

This approximation was discussed in detail by Calzetta and Hu[27], and recently utilized by Berges and Cox[14] to discuss equilibration in 1+1 dimensional quantum field theory. The approximation, which we denote as 2-PI, is lower order than the BVA approximation in that it can be obtained from the BVA by keeping only the *first* term in the equation for  $\bar{D}(x, x')$  as a power series in  $\lambda$ . That is, if we approximate

$$\bar{D}(x, x') \approx -\lambda^2 \Pi(x, x') = -\lambda^2 G(x, x') G(x', x), \quad (79)$$

in the BVA, then

$$\bar{\Sigma}(x, x') \approx \frac{\lambda^2}{2} [G(x, x')]^3. \quad (80)$$

So (55) reduces to:

$$[\square + \bar{\chi}(t)] G(x, x') = \delta_C(x, x') - \frac{\lambda^2}{2} \int_C d^2 x'' [G(x, x'')]^3 G(x'', x'). \quad (81)$$

We can rewrite this as:

$$\begin{aligned} & \left\{ \square + \mu^2 + \frac{3\lambda}{2} G_{>}(x, x)/i \right\} G_{>}(x, x') \\ &= -\frac{\lambda^2}{2} \int dx'' \left\{ \int_0^{t'} dt'' G_{>}(x', x'') G_{>}^3(x, x'') + \int_{t'}^t dt'' G_{>}(x'', x') G_{>}^3(x, x'') \right. \\ & \quad \left. - \int_0^t dt'' G_{>}(x'', x') G_{>}^3(x'', x) \right\}, \quad (82) \end{aligned}$$

which agrees with Eqs. (9) and (10) of Berges and Cox[14] apart from a symmetry factor of 1/3 on the right hand side of Eq. (82). For the classical case, we need to only include in  $\Pi_{>}(x, x')$  and  $\Pi_{<}(x, x')$  the subset of vertices that come from the classical MSR formalism (see appendix B). What we find is that the results for the 2-PI approximation are quite similar to those found for the BVA.

## VI. RESULTS

The present codes for evolving the SD equations use spectral numerical methods based on Chebyshev polynomial expansions, as explained in[28, 29]. Thus our calculations are not

lattice calculations in coordinate space using periodic boundary conditions, but are carried out entirely in momentum space. They take advantage of the global spectral convergence character of the Chebyshev polynomial expansion, which allow for an exact continuum limit. The calculation of the different modes decouple and an implementation via the message passing interface [MPI] on a parallel computer is straightforward.

We choose our initial parameters to match those given by Aarts, *et al.*[6]. We have taken  $\lambda = 1/3$  and  $\mu = 1$ , which are the values which make the classical problem parameter free. We set  $\Lambda = \pi/a = 4\pi\mu = 4\pi$ . This makes  $a = 1/4$ . The physical size of the system is then fixed by the value of  $N$  from the relation,  $L = Na = \pi N/\Lambda = N/4$ .

Also, to make contact with the simulations of Ref. [6], we choose our initial density matrix to be described by Eq. (11) with the mass parameter set equal to the bare mass  $\mu^2$ [33]. The value of  $\beta_0$  is taken to agree with the lattice coordinate space calculation of Ref. [6],  $1/\beta_0 = T_0 = 5.03891094$ , so that for the coordinate space Monte Carlo calculations,  $\langle \phi^2(x, 0) \rangle_0 = 2.5$  exactly. (This is for convenience only.)

For the Hartree and BVA approximations, we take:

$$f_k(0) = 1/\sqrt{2\omega_k}, \quad \dot{f}_k(0) = -i\omega_k f_k(0) = -i\sqrt{\omega_k/2}, \quad (83)$$

where  $\omega_k = \sqrt{k^2 + \chi(0)}$ , with

$$\chi(0) = \mu^2 + \frac{3\lambda}{2} \langle \phi^2(0) \rangle_0, \quad (84)$$

and  $I(\Lambda)$  is given by Eq. (30).

In Fig. 1, we show  $\phi_{\text{cl}}^2(t) \equiv \langle \phi^2(t) \rangle_0$  as a function of time, using the coordinate space lattice calculations of Eq. (25) together with the various approximations presented in this paper. The BVA, Hartree, large- $N$ , and coordinate space Monte Carlo codes use quite modest values of  $N = 128$  or  $256$ , as the calculation is very insensitive to this parameter. The momentum space Monte Carlo code uses somewhat larger values of  $N$  to obtain high accuracy.

We see that that the initial value  $\phi_{\text{cl}}^2(0)$  is different for the lattice coordinate and momentum space approaches, as explained in Section II D. For fixed values of the parameters  $\beta_0$  and  $\Lambda$ , the value of  $\phi_{\text{cl}}^2(0)$  is different in the two formalisms. One can of course force  $\langle \phi^2(0) \rangle_0$  to be the same, but then the mismatch for finite cutoff would appear somewhere else. Or one may try to correct this problem by shifting the lattice result such that it matches the

continuum approximations at  $t = 0$ . We show the results of such a shift in Fig. 2. However, such a naive solution has no fundamental support and may be misleading.

In order to compare directly our approximation methods to the exact Monte Carlo results, we have solved the classical Monte Carlo numerically in momentum space, as explained in Section IID. In Fig. 3, we illustrate these lattice momentum space results for  $\phi^2(t)$ , and compare them with the approximations presented here. The BVA result fares remarkably well and cures the oscillation problems of the Hartree approximation at large times. It does not display the re-amplifications (secularity) problems manifested by the result of Aarts *et al*, where the next-to-leading order evolution is obtained by truncating the effective action up to four-point couplings[6]. Similar behavior is noted for the time-evolution of  $\chi(t)$ , depicted in Fig. 4, and the *effective* temperature  $T_{\text{eff}} = \langle \pi^2(t) \rangle$ , shown in Fig. 5.

In Fig. 6, we compare the BVA result with the result of a calculation for which we truncate the  $D(x, x')$  equation at leading order, which makes it similar to the procedure advanced by Berges and Cox[14]. The results are very close indeed. The reason seems to be that the Berges approximation is expected to give reasonable results at short times and the differences should build up as the time-evolution progresses. However, at later times the thermalization mechanism takes over and virtually obliterates the differences. Since the classical problem scales, we have only computed results for  $\mu = 1$  and  $\lambda = 1/3$ . In the quantum case, a second dimensionfull parameter ( $\hbar$ ) enters, so that the problem no longer scales, and one will have to treat the case of large coupling constants ( $\lambda = 7.3$  for the linear sigma model) and high temperatures. Then the intermediate domain, between the regime when the Berges' leading order approximation for  $D(x, x')$  holds, and the regime when the thermalization blurs the differences, will become more important.

## VII. CONCLUSIONS

In this paper we have compared exact numerical simulations of classical scalar 1+1 dimensional field theory with several Schwinger-Dyson truncation schemes. We have focused on approximations that do not suffer from obvious problems, such as secularity and negative probability, by obtaining them from self-consistent approximations to the effective action which have been shown by Kraichnan to correspond to physically realizable systems. In particular, we have shown that a resummation of the  $1/N$  expansion that we studied earlier

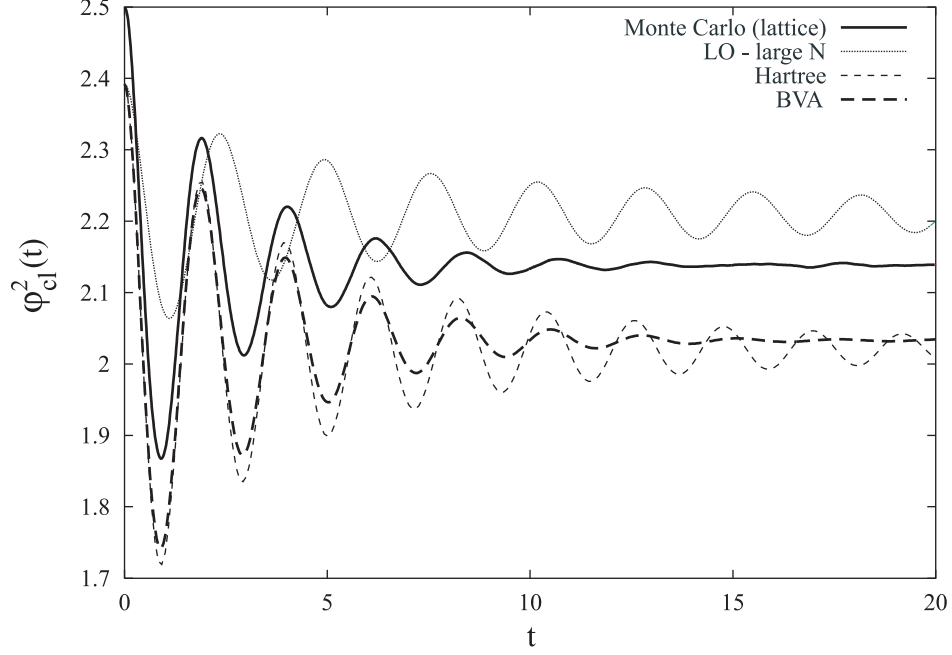


FIG. 1: The lattice coordinate space Monte Carlo calculation of  $\phi_{cl}^2(t) \equiv \langle \phi^2(t) \rangle_0$ , compared with the leading order large- $N$ , Hartree, and BVA approximations.

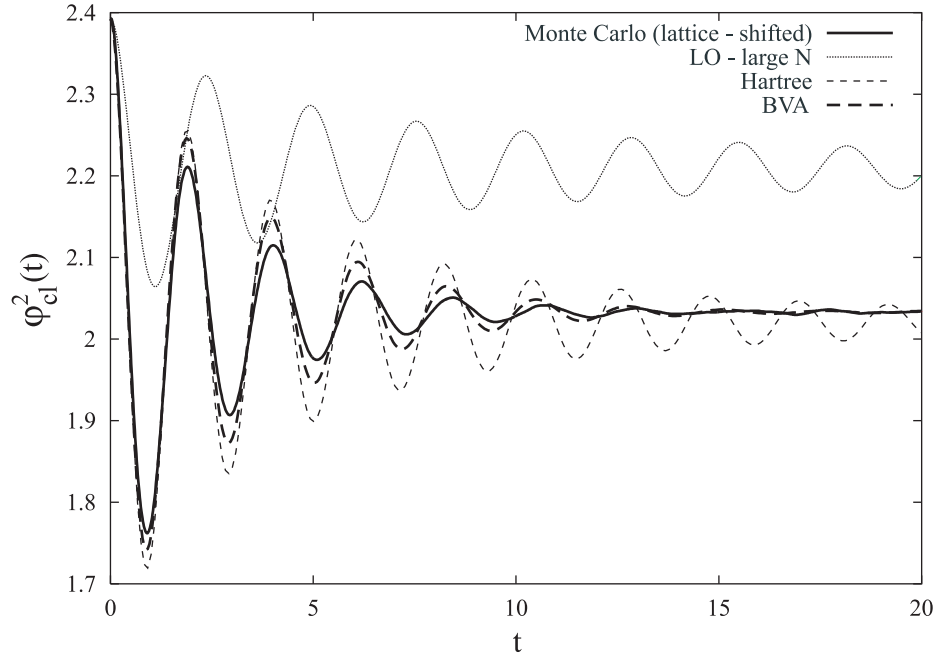


FIG. 2: The *shifted* lattice Monte Carlo calculation of  $\phi_{cl}^2(t) \equiv \langle \phi^2(t) \rangle_0$ , compared with the leading order large- $N$ , Hartree, and BVA approximations. Here, a constant factor is subtracted from the lattice coordinate space Monte Carlo results of Fig. 1.

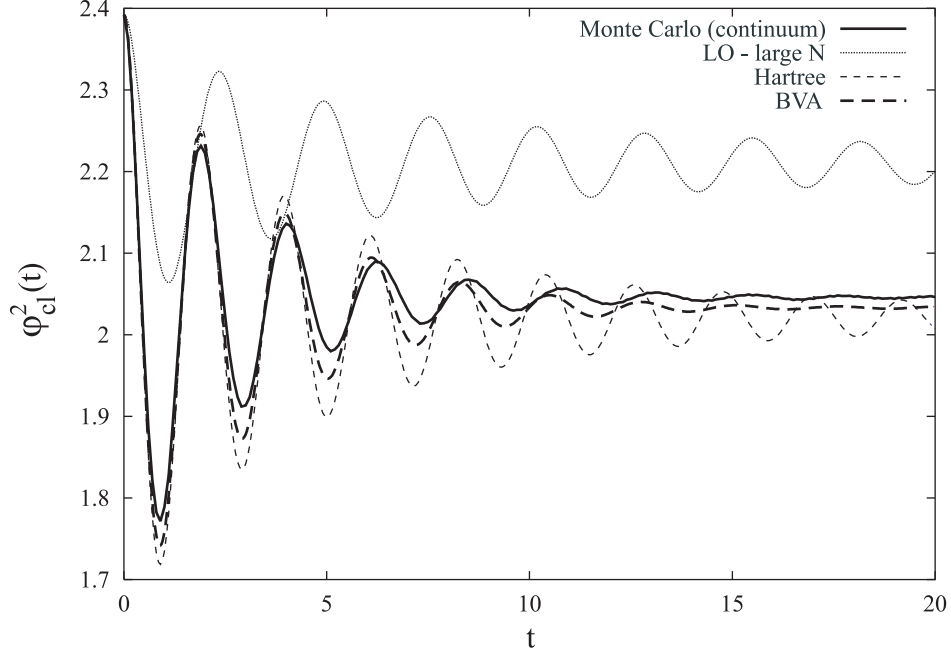


FIG. 3: The lattice momentum space Monte Carlo calculation of  $\phi_{cl}^2(t) \equiv \langle \phi^2(t) \rangle_0$ , compared with the leading order large- $N$ , Hartree, and BVA approximations.

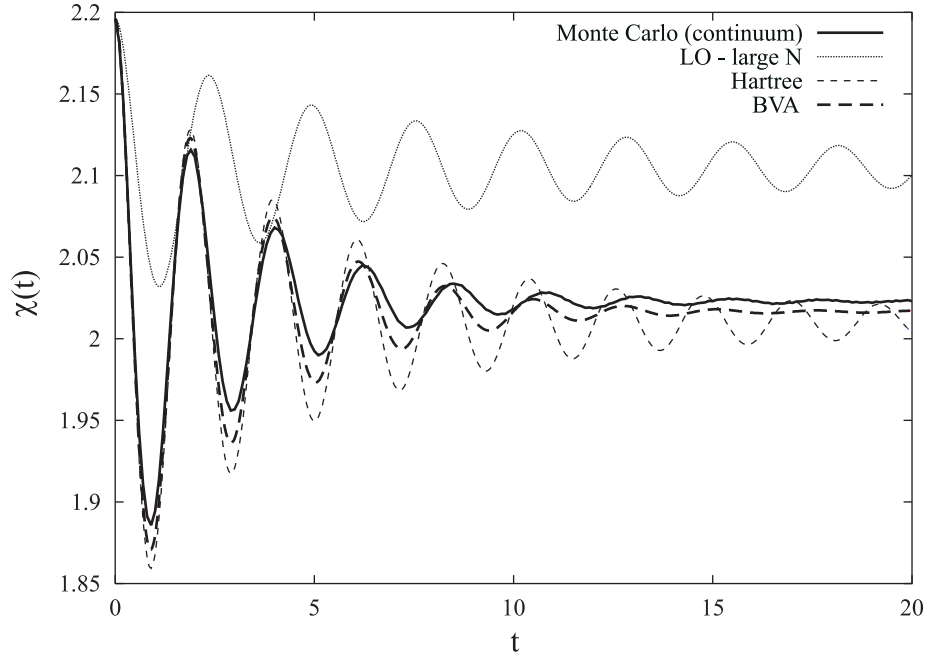


FIG. 4: The lattice momentum space Monte Carlo calculation of  $\chi(t)$ , compared with the leading order large- $N$ , Hartree, and BVA approximations.

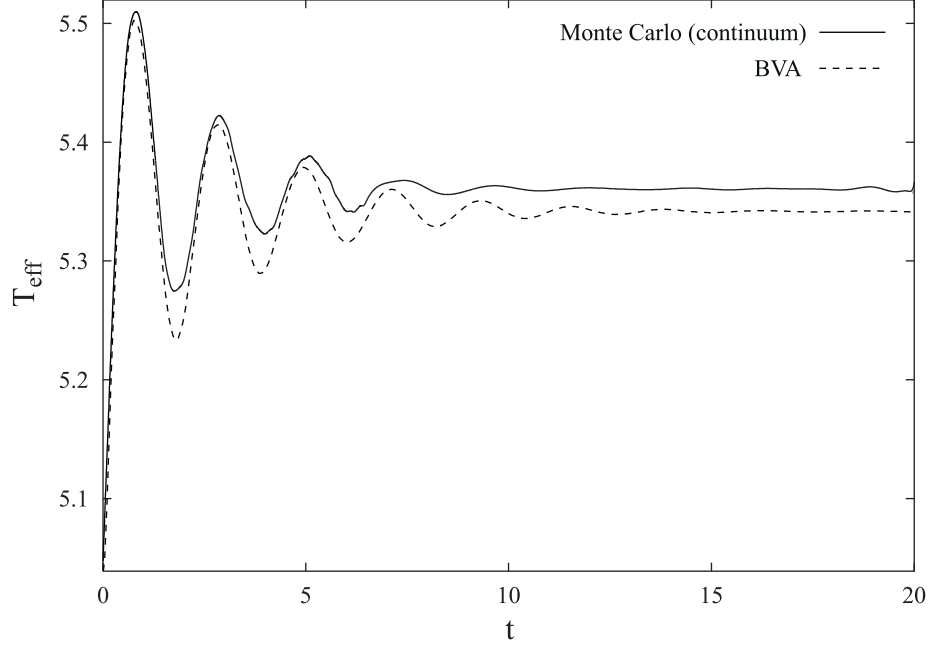


FIG. 5: The effective temperature,  $T_{eff}$ , as defined in the text, for the lattice momentum space Monte Carlo calculation and the BVA result.

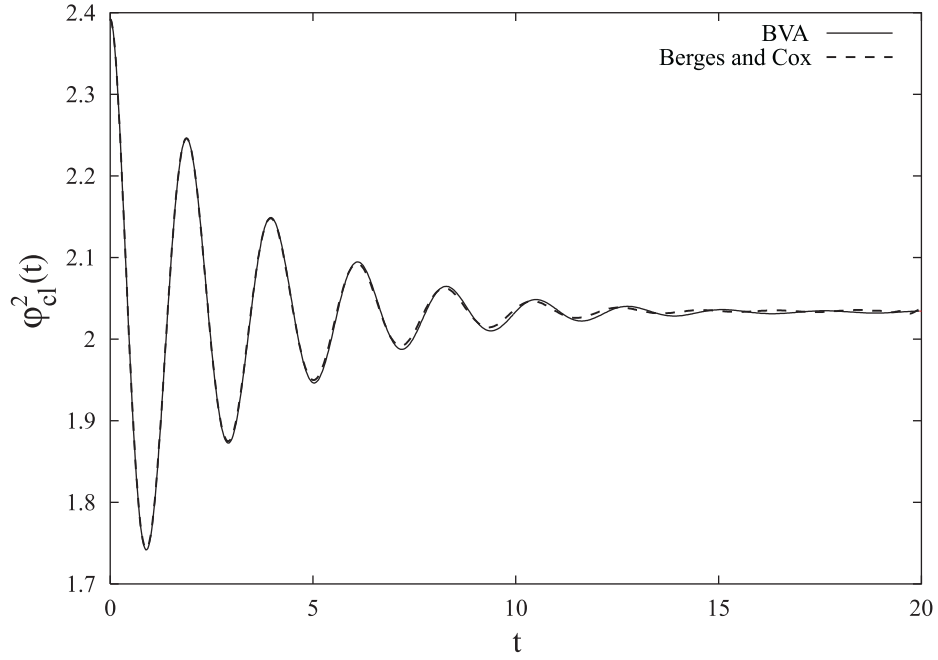


FIG. 6:  $\phi_{cl}^2(0) \equiv \langle \phi^2(t) \rangle_0$  for the BVA approximation and the approximation of Berges and Cox, for  $\mu = 1$ ,  $\lambda = 1/3$ . Note the close agreement.

in a quantum mechanics model[16], the bare vertex approximation [BVA], cures the late time oscillation problem of the Hartree approximation, and leads to results that quantitatively agree with exact numerical simulations, even at late times. Since the classical field theory, averaged over an initial Maxwell-Boltzmann distribution, is the high temperature limit of the quantum field theory, averaged over an initial Bose-Einstein distribution, our results imply that the BVA when applied to the quantum evolution meets the minimum requirement of being valid in the high temperature domain.

Our results give us confidence that this approximation will be useful in future studies of quantum phase transitions in the  $O(4)$  model. The fact that in homogeneous situations this approximation leads to thermalization, will allow us to study the rate of equilibration versus expansion rate for an expanding plasma undergoing a phase transition. We will then be able to see whether some of the interesting phenomena occurring during the early stages of a chiral phase transition will survive the hard scatterings present in the BVA approximation. At the same time this study was done, a complementary study of this approximation in the context of the quantum field theory in 1+1 dimension was performed with results not that different from the classical case[30]. Again, in the quantum simulations, the oscillations present in the Hartree approximation were found to damp, and equilibration occurred.

## APPENDIX A: CTP FORMALISM

For the quantum field theory problem, the SD equations for initial value problems can be obtained directly from the CTP path integral[19] for the generating functional. The functional differential equation for the generating functional is exactly the same as that of ordinary quantum field theory, with the exception that the  $\Theta$ -functions used in defining time-ordered products are defined on the closed-time path contour, rather than on the real line. This leads to a  $2 \times 2$  matrix formulation for the Green functions[19]. It is useful to have three different forms of the CTP formalism. The first is the original matrix formulation of Schwinger, the second uses closed contour  $\Theta$ -functions, and the third is a rewrite of the matrix formalism in terms of Wightman functions and advanced and retarded propagators.

The generating functional for initial value problem Green's functions is

$$Z[J^+, J^-] = e^{iW[J_\alpha]} = \int \prod_\alpha d\varphi_\alpha \exp i\{ S[\varphi_\alpha] + J_\alpha \varphi^\alpha \} \langle \varphi_1, i | \rho | \varphi_2, i \rangle, \quad (\text{A1})$$



where  $\langle \varphi_1, i | \rho | \varphi_2, i \rangle$  is the density matrix defining the initial state. We use the matrix notation

$$\varphi^a = \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}, \quad \text{for } a = 1, 2. \quad (\text{A2})$$

On this matrix space there is an indefinite metric

$$c_{ab} = \text{diag}(+1, -1) = c^{ab} \quad (\text{A3})$$

so that, for example,

$$J^a c_{ab} \varphi^b = J_+ \varphi_+ - J_- \varphi_- . \quad (\text{A4})$$

From the path integral we get the following matrix Green's function:

$$G^{ab}(t, t') = \frac{\delta^2 W}{\delta J_a(t) \delta J_b(t')} \Big|_{J=0}. \quad (\text{A5})$$

We can also write this using CTP  $\Theta$ -functions as

$$\begin{aligned} G(x, x') &= \text{Tr} \{ \mathcal{T}_C [ \hat{\phi}(x) \hat{\phi}(x') ] \} = \Theta_C(t, t') G_{>}(x, x') + \Theta_C^T(t', t) G_{<}(x, x'), \\ &= \begin{pmatrix} G_{++}(x, x') & G_{+-}(x, x') \\ G_{-+}(x, x') & G_{--}(x, x') \end{pmatrix}, \end{aligned} \quad (\text{A6})$$

where the CTP  $\Theta$ -functions are defined by:

$$\Theta_C(t, t') = \begin{cases} 1 & \text{for } t > t' \text{ on } \mathcal{C}, \\ 0 & \text{for } t < t' \text{ on } \mathcal{C}, \end{cases} \quad (\text{A7})$$

on the time contour  $\mathcal{C}$ . In matrix form, we have:

$$\Theta_C(t, t') = \begin{pmatrix} \Theta(t - t') & 0 \\ 1 & \Theta(t' - t) \end{pmatrix}, \quad \Theta_C^T(t', t) = \begin{pmatrix} \Theta(t' - t) & 1 \\ 0 & \Theta(t - t') \end{pmatrix}, \quad (\text{A8})$$

which leads to

$$\begin{aligned} G_{++}(x, x') &= \Theta(t - t') G_{>}(x, x') + \Theta(t' - t) G_{<}(x, x'), \\ G_{-+}(x, x') &= G_{>}(x, x') \\ G_{+-}(x, x') &= G_{<}(x, x') \\ G_{--}(x, x') &= \Theta(t' - t) G_{>}(x, x') + \Theta(t - t') G_{<}(x, x'), \end{aligned}$$

where

$$G_{>}(x, x')/i = \text{Tr}[\hat{\rho} \hat{\phi}(x) \hat{\phi}(x')] , \quad G_{<}(x, x')/i = \text{Tr}[\hat{\rho} \hat{\phi}(x') \hat{\phi}(x)] , \quad (\text{A9})$$

with  $\hat{\phi}(x)$  and  $\hat{\rho}$  quantum operators. Here,  $G_{\gtrless}(x, x')$  satisfy the symmetry relations

$$\left[ G_{\gtrless}(x, x')/i \right]^* = \left[ G_{\gtrless}(x', x)/i \right] = \left[ G_{\lesseqgtr}(x, x')/i \right] . \quad (\text{A10})$$

If we use the Schwinger-Keldysh basis then we define the  $\phi\phi\phi$  vertex as:

$$\frac{g}{3} h_{abc} \phi^a \phi^b \phi^c , \quad h_{abc} = \pm 1 , \quad \text{if } a = b = c = \pm 1. \quad (\text{A11})$$

The two point function SD equation can be written as

$$G^{ab} = G_0^{ab} - G_0^{ac} \Sigma_{cd} G^{db} . \quad (\text{A12})$$

To make a direct relation with classical correlation functions, it is convenient to go to another basis, given by

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} (\phi_+ + \phi_-)/2 \\ \phi_+ - \phi_- \end{pmatrix} = R \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \quad (\text{A13})$$

with

$$R = \begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix} . \quad (\text{A14})$$

To make contact with the MSR formalism discussed below, we will choose  $\phi_1 = \phi$  and  $\phi_2 = \hbar \hat{\pi}_\phi$ , so that

$$\phi_\pm = \phi \pm \frac{\hbar}{2} \hat{\pi}_\phi . \quad (\text{A15})$$

The usual matrix for the propagator

$$\mathbf{G}(x - x') = \begin{pmatrix} G_{++}(x - x') & G_{+-}(x - x') \\ G_{-+}(x - x') & G_{--}(x - x') \end{pmatrix} , \quad (\text{A16})$$

gets transformed into

$$\mathbf{G}(x - x') \rightarrow R \mathbf{G}(x - x') R^T = \begin{pmatrix} iF(x - x') & G_R(x - x') \\ G_A(x - x') & 0 \end{pmatrix} , \quad (\text{A17})$$

with

$$F(x-x') = \frac{1}{2} \langle \phi(x) \phi(x') + \phi(x') \phi(x) \rangle ,$$

$$G_R(x-x') = G_A(x'-x) = i \langle [\phi(x), \phi(x')] \rangle \Theta(t-t') . \quad (\text{A18})$$

The last equation can be used to define the spectral function:

$$\sigma(x-x') = i \langle [\phi(x), \phi(x')] \rangle = G_R(x-x') - G_A(x-x') . \quad (\text{A19})$$

In the advanced-retarded basis we then get the integral equations:

$$F(x, x') = F_0(x, x') - \iint dx'' dx''' \{ G_{0A}(x, x'') \Sigma_F(x'', x''') G_R(x''', x') \\ + F_0(x, x'') \Sigma_R(x'', x''') G_R(x''', x') + G_{0A}(x, x'') \Sigma_A(x'', x''') F(x''', x') \}$$

$$G_A(x, x') = G_{0A}(x, x') - \iint dx'' dx''' G_{0A}(x, x'') \Sigma_A(x'', x''') G_A(x''', x')$$

$$G_R(x, x') = G_{0R}(x, x') - \iint dx'' dx''' G_{0R}(x, x'') \Sigma_R(x'', x''') G_R(x''', x') . \quad (\text{A20})$$

The SD equations for our constrained  $\chi$  field problem can be obtained by considering first two propagating fields, and then taking the composite field limit for the second field, so the bare propagator for the second field is replaced by a delta function (see Cooper and Haymaker[31]). Since the second approach is more transparent in the MSR formalism we will follow that here. Thus we start with the Lagrangian

$$L = \frac{1}{2} \left\{ (\partial_\mu \phi)^2 + (\partial_\mu \chi)^2 - m^2 \phi^2 - M^2 \chi^2 - g \chi \phi^2 \right\} - j\phi - S\chi . \quad (\text{A21})$$

This leads to the equations of motion in the presence of external sources:

$$[\square + m^2] \phi(x) = -g \chi(x) \phi(x) - j(x) . \quad (\text{A22})$$

$$[\square + M^2] \chi(x) = -g \phi^2(x)/2 - S(x) . \quad (\text{A23})$$

After we determine the SD equations for the two field problem, we recover the original equations for the single  $\phi^4$  field theory by taking the composite limit where:

$$\chi(x) = \mu^2 + \frac{\lambda}{2} \phi^2(x)$$

$$M^2 = -\frac{1}{\lambda}, \quad g = 1, \quad S \rightarrow \frac{\mu^2}{\lambda},$$

$$D_{0R}(x-y) \rightarrow -\lambda \delta(x-y) . \quad (\text{A24})$$

We want to convert the Schwinger-Keldysh Lagrangian to the advanced retarded one. Starting from:

$$L = \frac{1}{2} \left\{ (\partial_\mu \phi_+)^2 - (\partial_\mu \phi_-)^2 + (\partial_\mu \chi_+)^2 - (\partial_\mu \chi_-)^2 - m^2 (\phi_+^2 - \phi_-^2) - M^2 (\chi_+^2 - \chi_-^2) - g (\chi_+ \phi_+^2 - \chi_- \phi_-^2) \right\} - j(\phi_+ - \phi_-) - S(\chi_+ - \chi_-). \quad (\text{A25})$$

we obtain after changing variables to  $\phi, \hat{\pi}_\phi$

$$\begin{aligned} -L_{\text{CTP}}/\hbar = & \hat{\pi}_\phi [\square + m^2] \phi + \hat{\pi}_\chi [\square + M^2] \chi + g \hat{\pi}_\phi \chi \phi + \frac{g}{2} \hat{\pi}_\chi \phi^2 \\ & + \frac{\hbar^2}{8} g \hat{\pi}_\phi^2 \hat{\pi}_\chi - j_\phi \phi - j_\chi \chi - j_{\hat{\pi}_\phi} \hat{\pi}_\phi - j_{\hat{\pi}_\chi} \hat{\pi}_\chi. \end{aligned} \quad (\text{A26})$$

We will find that we get exactly the same Lagrangian in the MSR formalism except that the term proportional to  $\hbar^2$  is missing.

## APPENDIX B: MSR FORMALISM

In the paper of Martin-Siggia-Rose [MSR][18], an operator formalism was developed which allowed them to find a generating functional for both the retarded and Wightman functions for first order classical field equations of the type:

$$\dot{x}(r, t) = A[x(r, t)] \quad (\text{B1})$$

where  $A[x(r, t)]$  is a local polynomial in the classical field  $x(r, t)$ . In the work of MSR,  $A[x(r, t)]$  could contain dissipative terms as well as prescribed noise terms. The formalism presented in MSR is first order in time derivatives and not apparently covariant. We have recently shown[20] that there is a covariant subset of the MSR equations in terms of which all the MSR Green's functions can be recovered. This subset can be derived from a second order Lagrangian formulation which can be related to the  $\hbar \rightarrow 0$  limit of the CTP formalism of Schwinger and Keldysh as mentioned above.

### 1. First order formalism

First let us review the first order formalism. For the statistical classical field evolutions of  $N$  interacting classical fields  $\phi_a$ ,  $a = 1, 2, \dots, N$  then  $x(r, t)$  is the  $2N$  component field

consisting of  $\phi_a$  and the canonical momentum  $\pi_a = \dot{\phi}_a$ .

$$x = \begin{pmatrix} \phi_a \\ \pi_a \end{pmatrix} \quad (\text{B2})$$

If, for example, we restrict ourselves to cubic interactions, then the vector  $A$  is of the form:

$$A_i = c_i(r, t) + d_{ij}x_j(r, t) + e_{ijk}x_j(r, t)x_k(r, t)/2 \quad (\text{B3})$$

where  $i = 1, \dots, 4$ . In the MSR formalism one then introduces the operator

$$\hat{x}(r, t) \equiv -\frac{\delta}{\delta x(r, t)}$$

such that the commutation rule  $[x(r, t), \hat{x}(r', t)] = \delta(r - r')$  is true. Then if we define an operator Hamiltonian via:

$$H(t) = \int dr' \hat{x}_i(r', t) A_i(r', t), \quad (\text{B4})$$

then the equations of motion can be written in the compact form

$$\dot{x}(r, t) = [x(r, t), H(t)] \quad (\text{B5})$$

For the commutator to be true at all times, one needs that  $\hat{x}$  satisfies

$$\frac{d\hat{x}(r, t)}{dt} = [\hat{x}(r, t), H(t)] \quad (\text{B6})$$

for consistency. Therefore  $\hat{x}(r, t)$  is a functional of  $x(r, 0)$  and  $\hat{x}(r, 0)$ . The formal solution to these equations is given by

$$\begin{aligned} x(t) &= U^{-1}(t, 0) x(0) U(t, 0) \\ \hat{x}(t) &= U^{-1}(t, 0) \hat{x}(0) U(t, 0) \end{aligned} \quad (\text{B7})$$

where

$$\begin{aligned} U^{-1}(t, t_0) &= \mathcal{T} \left\{ \exp \left[ - \int_{t_0}^t H(t') dt' \right] \right\}, \\ U(t, t_0) &= \mathcal{T}^* \left\{ \exp \left[ + \int_{t_0}^t H(t') dt' \right] \right\}. \end{aligned}$$

Here  $\mathcal{T}$  corresponds to the usual time ordered product operation. The meaning of the expectation value  $\langle x(t) \hat{x}(t') \rangle$  is as follows. Given an initial probability function  $P[x(0)]$ , then

$$\langle x(t) \hat{x}(t') \rangle = \int dx(0) x[t, x(0)] \hat{x}[t', x(0), \hat{x}(0)] P[x(0)]. \quad (\text{B8})$$

Wherever  $\hat{x}(0)$  appears, it is replaced by

$$\hat{x}(0) \rightarrow -\frac{d}{dx(0)}$$

and it acts on everything to the right. This definition of the extended averaging procedure has three important properties.

1. The average of a product of  $x$ 's agrees with the conventional definition.
2. The time dependence of  $\langle x(t) \hat{x}(t') \rangle$  is consistent with the above equations of motion.
3. The expectation value of a product of  $x$  and  $\hat{x}$  which has an  $\hat{x}$  to the left vanishes if  $P[x(0)]$  goes to zero fast enough at large  $|x|$ .

The last property is crucial for the tridiagonal form of the Green functions and follows from the fact that

$$\int_{-\infty}^{+\infty} \frac{d}{dx} [x^n P(x)] dx = 0, \quad \text{if:} \quad \lim_{|x| \rightarrow \infty} x^n P[x] = 0.$$

Thus in particular  $\langle \hat{x}(t') x(t) \rangle = 0$ .

The meaning of the *hatted* operators is understood in terms of the response to the system to an external source, as is discussed in Refs.[18] and [20].

## 2. Second order formalism

Let us first write the equations for the two coupled fields  $\chi$  and  $\phi$  in first order form. We have

$$\begin{aligned} \frac{\partial \chi}{\partial t} &= \pi_\chi, & \frac{\partial \pi_\chi}{\partial t} &= [\nabla^2 + m^2] \phi + g \chi \phi, \\ \frac{\partial \phi}{\partial t} &= \pi_\phi, & \frac{\partial \pi_\phi}{\partial t} &= [\nabla^2 + M^2] \chi + g \phi^2/2. \end{aligned} \quad (\text{B9})$$

The operator Hamiltonian which generates these equations is

$$\begin{aligned} H_{\text{MSR}} = \int dx \bigg\{ & \hat{\phi} \pi_\phi + \hat{\chi} \pi_\chi \\ & - \hat{\pi}_\phi \left\{ [\nabla^2 + m^2] \phi + g \chi \phi \right\} - \hat{\pi}_\chi \left\{ [\nabla^2 + M^2] \chi + g \phi^2/2 \right\} \bigg\} \quad (\text{B10}) \end{aligned}$$

Following the arguments of Ref. [20], and using the commutation relations, we find that the independent covariant second order equations (adding sources) are

$$\begin{aligned}
[\square + m^2] \phi + g \chi \phi &= j_\phi, \\
[\square + M^2] \chi + g \phi^2/2 &= j_\chi, \\
[\square + m^2] \hat{\pi}_\phi + g \chi \hat{\pi}_\phi + g \phi \hat{\pi}_\chi &= j_{\hat{\pi}_\phi}, \\
[\square + M^2] \hat{\pi}_\chi + g \phi \hat{\pi}_\phi &= j_{\hat{\pi}_\chi}.
\end{aligned} \tag{B11}$$

Here  $\hat{\pi}_\phi(x, 0)$  is treated as the operator  $\delta/\delta\phi(x, 0)$  when one averages over the initial probability function in phase space. These equations are derivable from the Lagrangian density:

$$\begin{aligned}
-\mathcal{L}_{\text{MSR}} = \hat{\pi}_\phi [\square + m^2] \phi + \hat{\pi}_\chi [\square + M^2] \chi + g \hat{\pi}_\phi \chi \phi + g \hat{\pi}_\chi \phi^2/2 \\
- j_\phi \phi - j_\chi \chi - j_{\hat{\pi}_\phi} \hat{\pi}_\phi - j_{\hat{\pi}_\chi} \hat{\pi}_\chi.
\end{aligned} \tag{B12}$$

which is identical to (A26), except that the terms proportional to  $\hbar^2$  are missing. This is analogous to what we showed for the quantum mechanics example in Ref. [20]. The extra bare vertices of the quantum field theory calculation are obtained from the term:

$$\frac{\hbar^2}{8} g \hat{\pi}_\phi^2 \hat{\pi}_\chi. \tag{B13}$$

### 3. The classical SD equations

The way one derives the SD equations from the action is identical for both classical and quantum field theory. Thus we obtain the same structure classically, but there are fewer vertex functions. We now derive the SD equations for a generic cubic self-interacting field theory whether classical or quantum. For  $N$  interacting scalar fields, we introduce the column vector  $\Phi_\alpha$  composed of  $\Phi_i^1 = \phi$  and  $\Phi_i^2 = \hat{\pi}_i$  where  $i = 1, 2, \dots, N$ . We also need the metric  $g_{\alpha\beta}$  which is just  $\sigma_{\alpha\beta}^1$  as far as connecting the  $\Phi^1$  and  $\Phi^2$  sectors. Then we can write generically for cubic interactions:

$$\mathcal{L} = \frac{1}{2} \Phi^\alpha D_{0\alpha\beta}^{-1} \Phi^\beta + \frac{1}{3} \gamma_{\alpha\beta\rho} \Phi^\alpha \Phi^\beta \Phi^\rho - J_\alpha \Phi^\alpha, \tag{B14}$$

where

$$D_{0\alpha\beta}^{-1}(x - y) = g_{\alpha\beta} [\square + m^2] \delta(x - y). \tag{B15}$$

The generating functional is formally

$$Z[J] = \langle \mathcal{T} \{ \exp[ J_\alpha \Phi^\alpha ] \} \rangle = e^{W[J]} ,$$

Defining the “classical” field  $\Phi^\alpha$  and the connected 2-point function  $W_2(1, 2)$  via

$$\Phi^\alpha(1) = \frac{\delta W}{\delta J_\alpha(1)} , \quad W_2(1, 2)_{\alpha\beta} = \frac{\delta \Phi_\alpha(1)}{\delta J_\beta(2)} ,$$

one has that in the presence of sources

$$\begin{aligned} D_{0\alpha\beta}^{-1} \Phi^\beta(1) + \gamma_{\alpha\beta\rho} \left[ \frac{\delta \Phi^\beta(1)}{\delta J_\rho(1)} + \Phi^\beta(1) \Phi^\rho(1) \right] &= J_\alpha(1) , \\ D_{0\alpha\beta}^{-1} W^{\beta\rho}(1, 2) + \gamma_{\alpha\beta\sigma} \left[ \frac{\delta W^{\beta\sigma}(1, 1)}{\delta J_\rho(2)} + \Phi^\beta(1) W^{\sigma\rho}(1, 2) + \Phi^\sigma(1) W^{\beta\rho}(1, 2) \right] &= \delta_\alpha^\rho \delta(1 - 2) . \end{aligned} \quad (\text{B16})$$

To obtain the SD equation for the inverse two point function we first use the connection between the connected 3-point function and the 1-PI vertex function[10] obtained by first Legendre transforming from  $J$  to  $\Phi$  and using the chain rule

$$\frac{\delta}{\delta J_\alpha(1)} = \int d2 W(1, 2)^{\alpha\beta} \frac{\delta}{\delta \Phi(2)^\beta} . \quad (\text{B17})$$

We find:

$$\begin{aligned} \langle \mathcal{T} \{ \Phi_\alpha(1) \Phi_\beta(2) \Phi_\rho(3) \} \rangle &= \frac{\delta W_{\alpha\beta}(1, 2)}{\delta J^\rho(3)} \\ &= \int d4 d5 d6 W_{\alpha\alpha'}(1, 4) W_{\beta\beta'}(2, 5) W_{\rho\rho'}(3, 6) \Gamma^{\alpha'\beta'\rho'}(4, 5, 6) , \end{aligned} \quad (\text{B18})$$

where the 1-PI three point function is defined by

$$\Gamma_{\alpha\beta\rho}(1, 2, 3) = \frac{\delta W_{\alpha\beta}^{-1}(1, 2)}{\delta \Phi^\rho(3)} . \quad (\text{B19})$$

The usual SD equation is obtained by multiplying Eq. (B16) on the right by  $W^{-1}$  to obtain

$$W_{\alpha\beta}^{-1}(1, 2) = D_{0\alpha\beta}^{-1}(1, 2) + \Sigma_{\alpha\beta}(1, 2) + 2 \gamma_{\alpha\beta\rho} \Phi^\rho(1) \delta(1 - 2) , \quad (\text{B20})$$

with

$$\Sigma_{\alpha\beta}(1, 2) = \int d3 d4 \gamma_{\alpha\rho\sigma} W^{\rho\lambda}(1, 3) W^{\sigma\nu}(1, 4) \Gamma(3, 4, 2)_{\lambda\nu\beta} , \quad (\text{B21})$$



and where

$$\Gamma_{\alpha\sigma\rho}(1, 2, 3) = 2 \gamma_{\alpha\sigma\rho} \delta(1-3) \delta(1-2) + \frac{\delta\Sigma_{\alpha\sigma}(1, 2)}{\delta\Phi^\rho(3)}. \quad (\text{B22})$$

However  $\Sigma$  is a proper self energy and can be related[9] to the effective action  $\Gamma_2[G]$  for the 2-PI graphs of the theory via

$$\Sigma_{\alpha\beta}(1, 2) = 2 \frac{\delta\Gamma_2[G]}{\delta G^{\alpha\beta}(1, 2)}, \quad (\text{B23})$$

so that it is just a function of the bare vertices and the full Green functions. Using

$$\frac{\delta\Sigma_{\alpha\beta}(1, 2)}{\delta\Phi_\rho(3)} = \int d5 d6 \frac{\delta\Sigma_{\alpha\beta}(1, 2)}{\delta G^{\sigma\lambda}(5, 6)} \frac{\delta G^{\sigma\lambda}(5, 6)}{\delta\Phi_\rho(3)}, \quad (\text{B24})$$

we find that

$$\begin{aligned} \Gamma_{\alpha\beta\nu}(1, 2, 3) &= 2 \gamma_{\alpha\beta\nu} \delta(1-2) \delta(1-3) \\ &\quad - \int d4 d5 d6 d7 \Gamma_{\alpha\rho\sigma}(1, 4, 5) G^{\rho\eta}(4, 6) G^{\sigma\lambda}(5, 7) H_{\eta\lambda;\beta\nu}(6, 7; 2, 3). \end{aligned} \quad (\text{B25})$$

The scattering kernel  $H$  is defined by

$$\frac{\delta\Sigma_{\alpha\beta}(1, 2)}{\delta G^{\sigma\lambda}(5, 6)} = H_{\alpha\beta\sigma\lambda}(1, 2; 5, 6).$$

Self consistent approximations[9] are determined by keeping a certain class of graphs in  $\Gamma_2[G]$ , the sum of all 2-PI graphs made from bare vertices and full propagators. For cubic interactions, the BVA is obtained by keeping the graph

$$\int d1 d2 \gamma_{ijk} G^{il}(1, 2) G^{jm}(1, 2) G^{kn}(1, 2) \gamma_{lmn},$$

which then leads to the self energy being the one loop diagram, and the scattering kernel being single particle exchange. By using the variational definitions of  $\Sigma$  and  $H$  one is guaranteed an internally consistent approximation. In the BVA, the integral equation for the correlation function is

$$G_{ab}(1, 2) = G_{0ab}(1, 2) + \int d3 d4 G_{0ai}(1, 3) \gamma_{ijk} G^{jl}(3, 4) G^{km}(3, 4) \gamma_{lmn} G_{nb}(4, 2). \quad (\text{B26})$$

Expanding (B26), we can write this in terms of classical and quantum contributions, as shown in [20]. We identify the self energy in this approximation as

$$\Sigma_{in}(3, 4) = \gamma_{ijk} G^{jl}(3, 4) G^{km}(3, 4) \gamma_{lmn}. \quad (\text{B27})$$

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